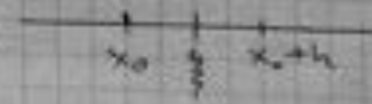
08.01.09 Analiza matematyczna wykład

... Szeregi Taylora $[x_0 ; x_0+h]$



f jest ciągła

f' jest ciągła na (x_0, x_0+h)

$$f(x_0+h) - f(x_0) = f'(\xi)\cdot h \qquad \xi \in (x_0, x_0+h)$$

f ma na przedziale $[x_0, x_0+h]$ n pochodnych to istnieje $\xi \in (x_0, x_0+h)$ t. że

$$f(x_0+h) = f(x_0) + \frac{f'(x_0)}{1!}h + \frac{f''(x_0)}{2!}h^2 + \ldots + \frac{f^{(n-1)}(x_0)}{(n-1)!}h^{n-1} + R_n(h, \xi)$$

Postać Lagrange'a: $R_n(h,\xi) = \frac{f^{(n)}(\xi)}{n!}h^n$

Postać Peano: $R_n = o(h^n)$ $\qquad \frac{R_n}{h^n} \to 0$

$f = o(g)$ w punkcie $x_0 \Leftrightarrow \lim_{x \to x_0} \frac{f(x)}{g(x)} = 0$

$f = O(g)$ w punkcie $x_0 \Leftrightarrow$ w otoczeniu x_0 $\left|\frac{f(x)}{g(x)}\right| \leq M$

np: $\sin x = O(x)$ w otoczeniu 0

$\lim \frac{\sin x}{x} = 1$

$\sin x = o(\sqrt{x})$

$\ln x^2 = o\left(\frac{1}{x}\right)$ $\qquad \frac{\ln x^2}{\frac{1}{x}} = x \ln x^2 \xrightarrow[x\to 0]{} 0$

6. $f(x)=\ln\frac{1+x}{1-x}$ $|x|<1$ $x_0=0$ $\frac{1+h}{1-h}=p>0$ $f(x_0+h)=f(h)=\ln\frac{1+h}{1-h}=2h+2\frac{h^3}{3}+2\frac{h^5}{5}+\dots$ $h=\frac{p-1}{p+1}$ $\ln(10)\sim\frac{10-1}{10+1}=\frac{9}{11}$

Przykład: 1 $f(x)=e^x$ $x_0=0$ $h\in\{0\}$

$f^{(n)}(x)=e^x$ $e^{0+h}=e^h=1+\frac{1}{1!}h+\frac{1}{2!}h^2+\frac{1}{3!}h^3+\dots+\frac{1}{(n-1)!}h^{n-1}$

$f^{(n)}(0)=e^0=1$ $e^h\approx 1+h+\frac{h^2}{2}+\frac{h^3}{6}+\frac{h^4}{24}$ $+\frac{e^\xi}{n!}h^n$

$e=1+1+\frac{1}{2}+\frac{1}{6}+\frac{1}{24}$ $+o(h^n)$

$|R_n(\xi,h)|=\frac{e^\xi}{120}<\frac{3}{120}$

x_0 — ξ — x_0+h

błąd $<\frac{1}{1000}$ $\frac{3}{n!}<\frac{1}{1000}$

$e^2=1+2+\frac{4}{2}+\frac{8}{6}+\frac{16}{24}+\frac{32}{120}+\dots$

$f(x)=\sin x$ $x_0=0$ 2. Rozwiń $\sin x$ w szereg.

$f'(x)=\cos x$ $f'(0)=1$

$f''(x)=-\sin x$ $f''(0)=0$

$f'''(x)=-\cos x$ $f'''(0)=-1$

$f''''(x)=\sin x$ $f''''(0)=0$

sin nieparzyste potęgi rozwinięcia Taylora („f. nieparzysta")

$\sin h=\frac{1}{1!}h-\frac{1}{3!}h^3+\frac{1}{5!}h^5-\frac{1}{7!}h^7+\frac{1}{9!}h^9+\dots$

$\cos h=1-\frac{h^2}{2!}+\frac{h^4}{4!}-\frac{h^6}{6!}+\frac{h^8}{8!}+\dots$

$\pm\frac{\sin\xi/\cos\xi}{n!}h^n$

$|R_n|\leq\frac{1}{n!}|h|^n$

3. $f(x)=\sqrt{x}$ $x_0=4$ $f(x)=x^{1/2}$ $f'(x)=\frac{1}{2}x^{-1/2}$

$f''(x)=\frac{1}{2}\left(-\frac{1}{2}\right)x^{-3/2}$ $f'''(x)=\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)x^{-5/2}$

$f^{(n)}(x)=(-1)^{n-1}\frac{1\cdot 3\cdot\ldots\cdot(2n-3)}{2^n}x^{-(2n-1)/2}$

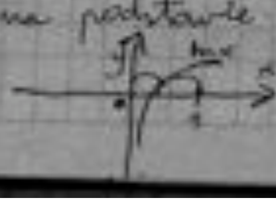
4. $f(x)=\frac{1}{(\sqrt{x})^3}$

5. $\ln(1+x)$; $x_0=0$

$f(x_0+h)=\ln(1+h)=h-\frac{h^2}{2}+\frac{h^3}{3}-\frac{h^4}{4}+\dots$

dla $h=1$ ciąg nie zbieżny na podstawie k. Leibniza:

$a_n=(-1)^{n-1}\frac{2^n}{n}$



08.01.09 Zastosowanie rachunku pochodnych do wyznaczania szeregów Taylora (Mclaurina) $\to x_0 = 0$

$f(x,y) = 3x^2y - 2x\cos x + \sqrt{1+x^2+y^2}$ $\quad$ $f: R \times R \to R$

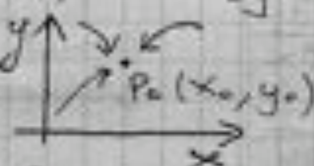
$f: R^n \to R^m$

$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \longrightarrow \begin{bmatrix} f_1(x_1 \dots x_n) \\ f_2(x_1 \dots x_n) \\ \vdots \\ f_m(x_1 \dots x_n) \end{bmatrix}$ np: $\begin{bmatrix} x \\ y \\ z \end{bmatrix} \to \begin{bmatrix} x^2+y^2 \\ 2xz+\sqrt{x^2+z^2+6} \end{bmatrix}$

$z = f(x,y) \qquad y = f(x)$

$\lim f(x,y) = g \iff$

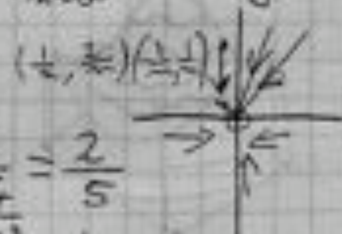
Dla każdego ciągu punktów $P_n = (x_n, y_n)$, gdzie $P_0 = (x_0, y_0)$ i $P_n \to P_0$, to ciąg wartości funkcji $f(x_n, y_n) = z_n \to g$



$f(x,y) = \frac{xy}{x^2+y^2} \qquad D_f = \mathbb{R}^3 - \{(0,0)\}$

$P_n = (\frac{1}{n}, \frac{1}{n})$ więc $f(\frac{1}{n}, 0) = \frac{\frac{1}{n}\cdot 0}{\frac{1}{n^2}+0} = 0 \underset{n\to\infty}{\to} 0$ czyli

$f(0, \frac{1}{n}) \to 0$ a



$f(\frac{1}{n}, \frac{1}{n}) = \frac{\frac{1}{n^2}}{\frac{1}{n^2}+\frac{1}{n^2}} = \frac{1}{2} \to \frac{1}{2} \qquad \frac{\frac{2}{n^2}}{\frac{1}{n^2}+\frac{4}{n^2}} = \frac{2}{5}$

$0 \le \left|\frac{x^2y}{x^2+y^2}\right| \le \left|\frac{x^2y}{x^2}\right| = |y|$

$a^2+b^2=1$

$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} + \begin{bmatrix} a \\ b \end{bmatrix} t$

$f(x,y) = f(x_0 + at, y_0 + bt) = \tilde{f}(t)$

$a = 0$
$b = 0$

$x(t) = x_0 + ta$

$y(t) = y_0 + tb$

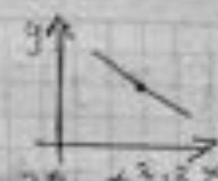
strzępizna

∇ ∂ Pochodne cząstkowe (w kierunkach osi)

Pochodna w punkcie Gradient (kierunek wzrostu f.)

$f(x,y) = xy^2$ $x(t) = 1 + \frac{t}{\sqrt{2}}$

$P = (1,1)$ $\vec{e} = \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$ $y(t) = 1 - \frac{t}{\sqrt{2}}$



$$f(x(t), y(t)) = \left(1+\frac{t}{\sqrt{2}}\right)\left(1-\frac{t}{\sqrt{2}}\right)^2 = \left(1+\frac{t}{\sqrt{2}}\right)\left(1-\frac{2t}{\sqrt{2}}+\frac{t^2}{2}\right) =$$

$$1 - \frac{2t}{\sqrt{2}} + \frac{t^2}{2} + \frac{t}{\sqrt{2}} - \frac{2t^2}{2} + \frac{t^3}{2\sqrt{2}} = 1 - \frac{t}{\sqrt{2}} - \frac{t^2}{2} + \frac{t^3}{2\sqrt{2}}$$

$$\tilde{f}'(t) = -\frac{1}{\sqrt{2}} - t + \frac{3}{2\sqrt{2}}t^2 = -\frac{1}{\sqrt{2}}$$

$D_{\vec{e}} f(x_0, y_0)$ $f(x,y) = xy^2$ $\frac{\partial f}{\partial x} = y^2$ $\frac{\partial f}{\partial y} = 2xy$

$\frac{\partial^2 f}{\partial x^2} = 0$ $\frac{\partial^2 f}{\partial y^2} = 2x$ $\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = 2y$ $\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = 2y$

$$\nabla f(x,y) = \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{bmatrix}$$

punkt siodłowy

$\frac{\partial f}{\partial x} = 0$

$\frac{\partial f}{\partial y} = 0$

$$\nabla f = \begin{bmatrix} y^2 \\ 2xy \end{bmatrix} \quad \nabla f(1,1) = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad \nabla f(0,1) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$D_{\vec{e}}(x_0, y_0) = \nabla f(x_0, y_0) \cdot \vec{e} = a\frac{\partial f}{\partial x} + b\frac{\partial f}{\partial y}$$

Szukanie ekstremów:

1. $\nabla f = 0$ Funkcja $f(x,y)$ ma w punkcie (x_0, y_0) lokalne ekstremum gdy spełnione są następujące warunki

1. $\nabla f(x_0, y_0) = 0$

2. $$\det \begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial y^2} \end{bmatrix}(x_0, y_0) > 0$$

→ znak dodatni maksimum ujemny minimum (wystarczy 1 element)

Wówczas jeśli $\frac{\partial^2 f}{\partial x^2}(x_0, y_0) > 0$ – minimum lub $\frac{\partial^2 f}{\partial x^2}(x_0, y_0) < 0$ – maximum

$$\nabla f(m,s) = \begin{bmatrix} \frac{\partial f}{\partial m} \\ \frac{\partial f}{\partial s} \end{bmatrix}$$

$\nabla \equiv 0 \Rightarrow$ nie ma nachylenia

Ponadto jeśli $\det(\ldots)_{(x_0, y_0)}$ < 0 - to nie jest to ekstremum

$= 0$ - pochodne wyższych rz.

Przykłady:

$f(x,y) = x^2 - y^2$

$\frac{\partial f}{\partial y} = 2x = 0 \Rightarrow (0,0)$ $\qquad$ $\frac{\partial f}{\partial y} = 2x = 0$ $\qquad$ $(0,0)$

$\frac{\partial f}{\partial y} = 2y = 0$ $\qquad$ $\frac{\partial f}{\partial y} = -2y = 0$

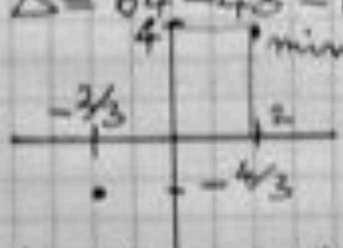
$\det \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}_{(0,0)} = 4 > 0$ $\qquad$ $\det \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix} = -4 < 0$

$f(x,y) = x^2 - 4xy + y^3 + 4y$

$\frac{\partial f}{\partial x} = 2x - 4y = 0 \rightarrow x = 2y$

$\frac{\partial f}{\partial y} = -4x + 3y^2 + 4 = 0 \rightarrow 3y^2 - 8y + 4 = 0$

$\Delta = 64 - 48 = 16 \quad \sqrt{\Delta} = 4 \quad y_{1,2} = \frac{8 \pm 4}{6}$ $\begin{cases} 2 \rightarrow x = 4 \\ -\frac{2}{3} \rightarrow x = -\frac{4}{3} \end{cases}$



$\det \begin{bmatrix} 2 & -4 \\ -4 & 6y \end{bmatrix}$ dla

$(x=4 \quad y=2)$ $\det \begin{bmatrix} 2 & -4 \\ -4 & 12 \end{bmatrix} = 48 - 16 > 0 \Rightarrow$ minimum

dla $(x = -\frac{4}{3}, y = -\frac{2}{3})$ $\det \begin{bmatrix} 2 & -4 \\ -4 & -4 \end{bmatrix} = -8 - 16 < 0$